Survey of India.

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THEORY OF ERRORS OF OBSERVATION.

PREPARED FOR USE IN THE DEPARTMENT

BY

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NOTES

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THEORY OF ERRORS OF OBSERVATION.

Let a stone be dropped with the intention that it shall strike a mark on the ground. Through the mark draw two lines at right angles and take these as axes of co-ordinates x and y.

Let $\phi(x)\,dx$ be the chance of the stone falling between the distances x and $x + dx$ from the axis of y.

then $\phi(y)\,dy$ will be the chance of the stone falling between the distances y and $y + dy$ from the axis of x.

Regarding these as independent events, the chance that the stone will fall on the small rectangle $dx\,dy$ is

$$\phi(x)\,\phi(y)\,dx\,dy.$$

For if p is the probability of an event which may happen in a ways and fail in b ways

and " q " another independent event a' " b' "

then $$p = \frac{a}{a+b} \quad \text{and} \quad q = \frac{a'}{a'+b'},$$

also the two events may happen in $a\,a'$ ways out of a total of $(a + b)\,(a' + b')$ ways,

therefore the probability of both happening $= \dfrac{a\,a'}{(a+b)\,(a'+b')} = p\,q.$

The chance of the stone falling on the small rectangle is therefore

$$\phi(x)\,\phi(y)\,d\sigma$$

where $d\sigma$ is an element of area about the point xy.

Now this must be independent of the direction in which the axes are drawn so that if we take a new set of axes, one through the mark and the point xy and the

other through the mark at right angles to this, the new co-ordinates of the point xy will be $\sqrt{x^2 + y^2}$ and 0,
and the chance of the stone falling on the small element of area $d\sigma$ will be

$$\phi \left\{ \sqrt{x^2 + y^2} \right\} \phi (0)\, d\sigma.$$

Therefore $$\phi (x)\ \phi (y) = \phi \left\{ \sqrt{x^2 + y^2} \right\} \phi (0).$$

Differentiate with regard to x and then with regard to y, and we get

$$\phi' (x)\ \phi (y) = \frac{x\, \phi' \left\{ \sqrt{x^2 + y^2} \right\} \phi (0)}{\sqrt{x^2 + y^2}},$$

$$\phi (x)\, \phi' (y) = \frac{y\, \phi' \left\{ \sqrt{x^2 + y^2} \right\} \phi (0)}{\sqrt{x^2 + y^2}}.$$

Dividing we get

$$\frac{\phi' (x)}{x\, \phi (x)} = \frac{\phi' (y)}{y\, \phi (y)}$$

so that $$\frac{\phi' (x)}{x\, \phi (x)} \text{ is constant.}$$

Put $$\frac{\phi' (x)}{x\, \phi (x)} = 2\, m;$$

therefore $$\frac{\phi' (x)}{\phi (x)} = 2\, m\, x,$$

therefore $$\frac{d}{dx} \log \phi (x) = 2\, m\, x,$$

therefore $$\log \phi (x) = mx^2 + \text{constant};$$

whence $$\phi (x) = \mathrm{C}\, e^{mx^2}.$$

Now the chance of the stone hitting a point xy must diminish as the point recedes from the mark, therefore $\phi (x)$ must diminish as x increases; so we may put $m = -\frac{1}{c^2}$,

therefore $$\phi (x) = \mathrm{C}\, e^{-\frac{x^2}{c^2}}.$$

Thus the chance of the stone hitting the ground between the lines x and $x + dx$ is

$$\phi (x)\, dx = \mathrm{C}\, e^{-\frac{x^2}{c^2}}\, dx.$$

The integral of this between the limits for x of $-$ infinity and $+$ infinity gives the chance of the stone hitting the ground somewhere and as this is certainty we may put

$$\mathrm{C} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{c^2}}\, dx = 1.$$

Put $$u = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{c^2}}\, dx,$$

As it is quite immaterial whether we use the letter x or the letter y in this integral,

we have $$u = \int_{-\infty}^{+\infty} e^{-\frac{y^2}{c^2}} dy,$$

therefore $$u^2 = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{c^2}} dx \times \int_{-\infty}^{+\infty} e^{-\frac{y^2}{c^2}} dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{c^2}-\frac{y^2}{c^2}} dx\, dy.$$

Now changing to polar co-ordinates, the elementary area $dx\,dy$ becomes $r\,d\theta\,dr$ and $x^2 + y^2 = r^2$.

Also since the integral extends to infinity in every direction, the limits for r, are 0 and ∞ and for θ, 0 and 2π.

Therefore $$u^2 = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{c^2}} r\,d\theta\,dr$$

$$= \pi \int_0^{\infty} e^{-\frac{r^2}{c^2}} d\,(r^2) \quad = c^2\pi \int_0^{\infty} e^{-\frac{r^2}{c^2}} d\left(\frac{r^2}{c^2}\right)$$

$$= c^2\pi$$

then $$u = c\sqrt{\pi},$$

therefore $$C c \sqrt{\pi} = 1.$$

Therefore the chance of the stone hitting the ground between the lines x and $x + dx$ is

$$\frac{1}{c\sqrt{\pi}} e^{-\frac{x^2}{c^2}} dx,$$

which therefore represents the probability of an error lying between x and $x + dx$.

Suppose that the total number of measures of a quantity is A, where A is a very large number, then we may expect the number of errors which fall between x and $x + \delta x$ to be $\left(\frac{A}{c\sqrt{\pi}} e^{-\frac{x^2}{c^2}} \delta x\right)$ where c is a modulus, constant for any one system of measures but different for different systems. This applies equally to positive (+) and negative (−) errors, the number A including all the measures whether the errors are + or −, and the number of + and − errors being practically identical when A is large.

I.—*Mean Error.*

Suppose now that the true value of the quantity is known so that each of all the errors can be found. Then take the mean of all the positive errors and also that of all the negative errors and the mean of these two without regard to sign.

This is called the *Mean Error* and is to be regarded as a number without a sign.

Since the number of errors whose magnitude is included between x and $x + \delta x$ $\Big\} = \dfrac{A}{c\sqrt{\pi}} e^{-\frac{x^2}{c^2}} \delta x$,

and the magnitude of each error does not differ much from x, so that the sum of these errors will be

$$\frac{A}{c\sqrt{\pi}} e^{-\frac{x^2}{c^2}} x\, \delta x.$$

Therefore sum of all errors of + sign $= \dfrac{A}{c\sqrt{\pi}} \displaystyle\int_0^\infty \delta x\, e^{-\frac{x^2}{c^2}} x = \dfrac{cA}{2\sqrt{\pi}}$,

and the number of all the errors of + sign $= \dfrac{A}{c\sqrt{\pi}} \displaystyle\int_0^\infty e^{-\frac{x^2}{c^2}} \delta x = \dfrac{A}{2}$;

therefore mean positive error $= \dfrac{c}{\sqrt{\pi}}$.

Similarly mean negative error $= \dfrac{c}{\sqrt{\pi}}$

Therefore *Mean error* $= \dfrac{c}{\sqrt{\pi}} = c \times 0{\cdot}564189$.

II.—*Error of Mean Square.*

Square each of the errors, take the mean of these squares and then extract its square root. This is called the *Error of Mean Square,* and is also a numerical quantity without sign.

Now, as before, the number of errors lying between x and $x + \delta x$ $\Big\} = \dfrac{A}{c\sqrt{\pi}} e^{-\frac{x^2}{c^2}} \delta x$

so that the sum of the squares of errors between x and $x + \delta x$ $\Big\} = \dfrac{A}{c\sqrt{\pi}} e^{-\frac{x^2}{c^2}} x^2\, \delta x$.

Therefore the sum of the squares of all errors $= \dfrac{A}{c\sqrt{\pi}} \displaystyle\int_{-\infty}^{+\infty} \delta x\, e^{-\frac{x^2}{c^2}} x^2$

$$= \left[-\frac{Ac}{2\sqrt{\pi}} x\, e^{-\frac{x^2}{c^2}}\right]_{-\infty}^{+\infty} + \frac{Ac}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \delta x\, e^{-\frac{x^2}{c^2}}$$

$$= \quad 0 \quad + \frac{Ac^2}{2} \quad = \frac{Ac^2}{2}.$$

But total number of errors $= A$,

therefore mean square = mean of squares $= \frac{Ac^2}{2} \div A = \frac{c^2}{2}$.

Therefore *Error of Mean Square* $= \sqrt{\frac{c^2}{2}} = c \times 0\cdot707107$.

III.—*Probable Error.*

By this is *not* meant that the number used is more probable than any other, but that if the + sign be used the number of errors greater than the probable error, is the same as the number of errors less than the probable error, and when the − sign is used the same remark applies to the negative errors. The probable error itself is a numerical quantity without sign.

Now the number of positive errors up to the value of $x = \frac{A}{c\sqrt{\pi}} \int_0^x \delta x \, e^{-\frac{x^2}{c^2}}$,

and the whole number of positive errors . . . $= \frac{A}{2}$,

therefore half the number of positive errors . . . $= \frac{A}{4}$.

Therefore to find the probable error we must put $\frac{A}{c\sqrt{\pi}} \int_0^x \delta x \, e^{-\frac{x^2}{c^2}} = \frac{A}{4}$,

and writing cw for x we must put

$$\frac{1}{\sqrt{\pi}} \int_0^w \delta w \, e^{-w^2} = \tfrac{1}{4}.$$

A table has been constructed for this integral, and from it the value of w which satisfies this equation

is found to be $w = 0\cdot476948$,

that is $x = c \times 0\cdot476948$,

therefore *Probable Error* $= c \times 0\cdot476948$.

IV.—*Probable Error &c. of aX.*

Suppose that in different measures of a quantity X, the errors x_1, x_2, x_3, *etc.* have been made, then if in our investigation a quantity aX comes in by purely algebraical transformation and *not* by measuring X, a times, the values of $Y = aX$ derived from these different measures are affected by errors ax_1, ax_2, ax_3, *etc.*, the number remaining the same. So that if X is liable to any number of errors of magnitude x, $x + \delta x$ or anything between them, then Y is liable to the same number of errors of magnitudes $ax = y$ or $ax + a\delta x = y + \delta y$ or anything between them.

The number of errors of Y or aX whose magnitudes fall between y and $y + \delta y$ is the same as the number of errors of X between x and $x + \delta x$ and is

$$\frac{A}{c\sqrt{\pi}}\, e^{-\frac{x^2}{c^2}}\, \delta x\,,$$

and as $x = \dfrac{y}{a}$, this expression $= \dfrac{A}{ac\sqrt{\pi}}\, e^{-\frac{y^2}{a^2c^2}}\, \delta y.$

This is exactly the same form as before, so that we have

1. The law of frequency of error for aX is similar to that for X.

2. The modulus is ac.

Therefore *Mean Error* of $aX = ac \times \cdot 564189 = a \times$ *Mean Error* of X

Error of Mean Square of $aX = ac \times \cdot 707107 = a \times$ *e.m.s.* of X

Probable Error of $aX = ac \times \cdot 476948 = a \times$ *p.e.* of X

V.—*Probable Error &c. of the sum of a number of quantities such as aX.*

Again when two fallible determinations X and Y are added together algebraically to form a result Z, the law of frequency for each will be the same as for $(X + Y)$, but the modulus will be got from

Square of modulus of Z = square of modulus of X + square of modulus of Y.

For the probability of an error x in X is $\dfrac{1}{c\sqrt{\pi}}\, e^{-\frac{x^2}{c^2}}\, dx$

and " " y in Y is $\dfrac{1}{c_1\sqrt{\pi}}\, e^{-\frac{y^2}{c_1^2}}\, dy$

therefore the probability of the simultaneous occurrence of these two is $\dfrac{1}{c\,c_1\,\pi}\, e^{-\frac{x^2}{c^2}-\frac{y^2}{c_1^2}}\, dx\, dy.$

Now an error x in X and an error y in Y produce an error z in Z according to the relation

$$z = x + y,$$

and this relation can always be satisfied by combining any value of y with all the values of x ranging from $-\infty$ to $+\infty$.

The probability therefore of an error z in Z may be written

$$\frac{1}{c\,c_1\,\pi}\,dy\int_{-\infty}^{+\infty} e^{-\frac{x^2}{c^2}-\frac{y^2}{c_1^2}}\,dx.$$

But since y is independent of x $\qquad dz = dy.$

Therefore the probability of an error z in Z

$$= \frac{1}{c\,c_1\,\pi}\,dz\int_{-\infty}^{+\infty} e^{-\frac{x^2}{c^2}-\frac{1}{c_1^2}(z-x)^2}\,dx$$

$$= \frac{1}{c\,c_1\,\pi}\,e^{-\frac{z^2}{c^2+c_1^2}}\,dz\int_{-\infty}^{+\infty} e^{-\left(\frac{1}{c^2}+\frac{1}{c_1^2}\right)\left(x-\frac{c^2 z}{c^2+c_1^2}\right)^2}\,dx$$

$$= \frac{1}{c\,c_1\,\pi}\,e^{-\frac{z^2}{c^2+c_1^2}}\,dz \times \sqrt{\frac{c^2\,c_1^2}{c^2+c_1^2}}\sqrt{\pi}$$

$$= \frac{1}{\sqrt{c^2+c_1^2}\;\sqrt{\pi}}\,e^{-\frac{z^2}{c^2+c_1^2}}\,dz.$$

$$= \frac{1}{c_2\sqrt{\pi}}\,e^{-\frac{z^2}{c_2^2}}\,dz.$$

where $$c_2^2 = c^2 + c_1^2.$$

So that the law of error of Z is the same as that of X and Y and the square of the modulus of Z = square of modulus of X + square of modulus of Y.

From this it follows that

$$(m.e.\text{ of } Z)^2 = (m.e.\text{ of } X)^2 + (m.e.\text{ of } Y)^2$$

$$(e.m.s.\text{ of } Z)^2 = (e.m.s.\text{ of } X)^2 + (e.m.s.\text{ of } Y)^2$$

$$(p.e.\text{ of } Z)^2 = (p.e.\text{ of } X)^2 + (p.e.\text{ of } Y)^2$$

Also since Y is liable to + and − errors of the same magnitude in equal numbers, it follows $-Y$ is liable to the same errors as $+Y$.

Therefore $\qquad p.e.\text{ of } -Y = p.e.\text{ of } +Y$

Therefore if $\qquad W = X - Y = X + (-Y),$

$$(p.e.\text{ of } W)^2 = (p.e.\text{ of } X)^2 + (p.e.\text{ of } -Y)^2 = (p.e.\text{ of } X)^2 + (p.e.\text{ of } Y)^2$$

Therefore $\qquad \{m.e.\text{ of } (X \pm Y)\}^2 = (m.e.\text{ of } X)^2 + (m.e.\text{ of } Y)^2$

$$\{e.m.s.\text{ of } (X \pm Y)\}^2 = (e.m.s.\text{ of } X)^2 + (e.m.s.\text{ of } Y)^2$$

$$\{p.e.\text{ of } (X \pm Y)\}^2 = (p.e.\text{ of } X)^2 + (p.e.\text{ of } Y)^2$$

$$\text{Also } \{p.e. \text{ of } (kX + lY)\}^2 = (p.e. \text{ of } kX)^2 + (p.e. \text{ of } lY)^2$$

$$= (k \times p.e. \text{ of } X)^2 + (l \times p.e. \text{ of } Y)^2$$

$$= k^2 \times (p.e. \text{ of } X)^2 + l^2 \times (p.e. \text{ of } Y)^2.$$

Similarly for *m.e.* and *e.m.s.*

$$\text{Also } \{p.e. \text{ of } (R + S + T)\}^2 = \{p.e. \text{ of } (R + S)\}^2 + (p.e. \text{ of } T)^2$$

$$= (p.e. \text{ of } R)^2 + (p.e. \text{ of } S)^2 + (p.e. \text{ of } T)^2.$$

Similarly

$$\{p.e. \text{ of } (rR + sS + tT + \text{etc.})\}^2 = r^2.\,(p.e. \text{ of } R)^2 + s^2\,(p.e. \text{ of } S)^2 + t^2(p.e. \text{ of } T)^2 + \text{etc.}$$

Similarly for *m.e.* and *e.m.s.*

VI.—*The Probable Error &c. of the sum of a number of different independent measures of the same physical quantity and of their mean.*

Let $X_1, X_2 \ldots X_n$ be all different independent measures of the same physical quantity or of equal physical quantities in every one of which the probable error is the same and equal to the *p.e.* of X_1.

$$\{p.e. \text{ of } (X_1 + X_2 + \ldots + X_n)\}^2 = (p.e. \text{ of } X_1)^2 + (p.e. \text{ of } X_2)^2 + \ldots + (p.e. \text{ of } X_n)^2$$

$$= n \times (p.e. \text{ of } X_1)^2,$$

$$\text{therefore} \qquad p.e. \text{ of } (X_1 + X_2 \ldots + X_n) = \sqrt{n}\ p.e. \text{ of } X_1.$$

Similarly for *e.m.s.*

Again

$$\text{Mean} = \frac{X_1}{n} + \frac{X_2}{n} + \ldots + \frac{X_n}{n},$$

$$\text{therefore } (p.e. \text{ of mean})^2 = \left(p.e. \text{ of } \frac{X_1}{n}\right)^2 + \left(p.e. \text{ of } \frac{X_2}{n}\right)^2 + \ldots + \left(p.e. \text{ of } \frac{X_n}{n}\right)^2$$

$$= \frac{1}{n^2}\left\{(p.e. \text{ of } X_1)^2 + (p.e. \text{ of } X_2)^2 + \ldots + (p.e. \text{ of } X_n)^2\right\}$$

$$= \frac{1}{n}\left(p.e. \text{ of } X_1\right)^2,$$

$$\text{therefore } p.e. \text{ of mean} = \frac{1}{\sqrt{n}}\ p.e. \text{ of } X_1.$$

Similarly for *e.m.s.*

VII.—*The Probable Error of a function ϕ (X, Y, Z, &c.) of one or several fallible quantities in terms of the Probable Error of each.*

It is supposed that the values x, y, z, &c. of the fallible quantities X, Y, Z, &c. are very approximately known, and therefore we may consider X equal to $x + \delta x$ where x is an absolute constant and δx a very small quantity liable to error, and where consequently the error of X is equal to the error of δx, and therefore the probable error of X is equal to the probable error of δx, and so for the others.

Now

$$\phi\,(X,\ Y, Z,\ \&c.) = \phi\,(x, y, z,\ \&c.) + \frac{d}{dx}\phi\,(x, y, z,\ \&c.)\ \delta x + \frac{d}{dy}\phi(x, y, z,\ \&c.)\ \delta y + \&c.$$

where everything is constant except δx, δy &c.

$$\text{Hence}\quad \Big\{p.e.\text{ of }\phi\,(X, Y, Z, \&c.)\Big\}^2 = \Big\{\frac{d}{dx}\phi\,(x,\ y,\ z, \&c.)\Big\}^2 \times\ (p.e.\text{ of }\delta x)^2$$

$$+\ \Big\{\frac{d}{dy}\phi\,(x, y, z, \&c.)\Big\}^2 + (p.e.\text{ of }\delta y)^2$$

$$+\ \&c.$$

Restoring the equivalents and remarking that the coefficients x, y, z, &c. are sensibly equal to X, Y, Z, &c.,

$$\Big\{p.e.\text{ of }\phi\,(X, Y, Z, \&c.)\Big\}^2 = \Big\{\frac{d}{dX}\phi\,(X,\ Y,\ Z,\ \&c.)\Big\}^2 \times\ (p.e.\text{ of }X)^2$$

$$+\ \Big\{\frac{d}{dY}\phi\,(X,\ Y, Z,\ \&c.)\Big\}^2 \times\ (p.e.\text{ of }Y)^2$$

$$+\ \&c.$$

Thus if x is the probable error of X,

then $$p.e.\text{ of }\log X = \frac{d}{dX}\log X \times x = \frac{x}{X}.$$

Again let $$W_1 = mX$$

and let μ be the $p.e.$ of m and x that of X,

then $$\log W_1 = \log m + \log X$$

therefore $$\{p.e.\text{ of }\log W_1\}^2 = \{p.e.\text{ of }\log m\}^2 + \{p.e.\text{ of }\log X\}^2$$

so that $$\frac{(p.e.\text{ of }W_1)^2}{{W_1}^2} = \frac{\mu^2}{m^2} + \frac{x^2}{X^2}.$$

Now let $$W_2 = \frac{m}{X}$$

therefore $$\log W_2 = \log m - \log X,$$

and similarly $$\frac{(p.e.\text{ of } W_2)^2}{W_2^2} = \frac{\mu^2}{m^2} + \frac{x^2}{X^2}$$

$$= \frac{(p.e.\text{ of } W_1)^2}{W_1^2}.$$

Therefore $$p.e.\text{ of } W_1 = p.e.\text{ of } W_2 \times X^2.$$

VIII.—*Probable Error &c. in a given series of observations.*

Suppose now that the true value of the quantity is not known. The mean of the observed values is taken to be the true value and it is afterwards proved in Section IX that this is the best value that can be adopted.

Suppose M to be the true value and a, b, c, etc. the errors. The observations then are $M + a$, $M + b$, $M + c$, etc.,

and the mean of the observations $= M + \frac{a + b + c + \&c.}{n}$

Therefore the apparent errors are

$$a - \frac{a + b + c + \&c.}{n}$$

$$b - \frac{a + b + c + \&c.}{n}$$

etc. etc.

Therefore the sum of squares of apparent errors

$$= a^2 - \frac{2a}{n}(a + b + c\ \&c.) + \frac{1}{n^2}(a + b + c + \&c.)^2$$

$$+ b^2 - \frac{2b}{n}(a + b + c\ \&c.) + \frac{1}{n^2}(a + b + c + \&c.)^2$$

$$+ c^2 - \quad \text{etc.} \quad \text{etc.}$$

$$= a^2 + b^2 + c^2 + \&c. - \frac{2}{n}(a + b + c + \&c.)^2 + \frac{n}{n^2}(a + b + c + \&c.)^2$$

$$= a^2 + b^2 + c^2 + \&c. - \frac{1}{n}(a + b + c + \&c.)^2.$$

But $(a + b + c + \&c.)^2 = a^2 + b^2 + c^2 + \&c.$
since there are as many $+ a$'s as there are $- a$'s and $+ b$'s as $- b$'s &c., so that the sum of all the products vanishes.

Therefore the sum of squares of apparent errors

$$= \frac{n-1}{n}(a^2 + b^2 + \&c.).$$

Now mean square of error of $a = a^2$, of $b = b^2$ etc. etc.

therefore the sum of squares of apparent errors

$$= \frac{n-1}{n} \text{(mean square of error of } a + \text{mean square of error of } b + \text{etc.)}$$

$$= \frac{n-1}{n}\{(\text{error of mean square of } a)^2 + (\text{error of mean square of } b)^2 + \dots \text{etc.}\}.$$

Then, taking error of mean square of a = error of mean square of b = etc. we get

Sum of squares of apparent errors

$$= (n-1)\{\text{error of mean square of a measure}\}^2.$$

Therefore error of mean square of a single measure

$$= \sqrt{\frac{\text{sum of squares of apparent errors}}{n-1}}$$

and error of mean square of the mean of the measures

$$= \sqrt{\frac{\text{sum of squares of apparent errors}}{n(n-1)}}$$

and probable error of a single measure

$$= 0{\cdot}6745 \sqrt{\frac{\text{sum of squares of apparent errors}}{n-1}}$$

and probable error of the mean of the measures

$$= 0{\cdot}6745 \sqrt{\frac{\text{sum of squares of apparent errors}}{n(n-1)}}.$$

IX.—*Combination of measures all equally good.*

Supposing that we had three measures of a quantity and that two of them were A and the third B. It is clear that a greater preponderance should be given to A than to B. In fact in taking the mean we should have :—

$$\frac{A + A + B}{3} = \frac{2A + B}{3}.$$

Following out this idea when we have different values of the quantity to be measured, and we are not sure that they are all of equal value, the method of combining them is to multiply each of them by a certain number called the *Combination Weight,* to add these results together and divide by the sum of the combination weights.

What combination weights are we to use?

They are to be determined so that the probable error of the result is to be as small as possible.

Suppose that we have n independent measures of a quantity all equally good so far as we can judge *à priori,* to find the proper method of combining them.

Let their *probable* errors be $e_1, e_2, e_3, \ldots e_n$ (each $= e$).

Let their *actual* errors be $E_1, E_2, E_3, \ldots E_n$ (these of course being unknown).

Let their combination weights be $w_1, w_2, w_3, \ldots w_n$.

Then the *actual* error of the result

$$= \frac{w_1 E_1 + w_2 E_2 + \ldots + w_n E_n}{w_1 + w_2 + \ldots + w_n}$$

$$= \frac{w_1}{w_1 + w_2 + \ldots} E_1 + \frac{w_2}{w_1 + w_2 + \ldots} E_2 + \ldots + \frac{w_n}{w_1 + w_2 + \ldots} E_n,$$

therefore (probable error of result)^2

$$= \left(\frac{w_1}{w_1 + w_2 + \ldots}\right)^2 e_1^2 + \left(\frac{w_2}{w_1 + w_2 + \ldots}\right)^2 e_2^2 + \text{etc.}$$

$$= \frac{w_1^2 e_1^2 + w_2^2 e_2^2 + \ldots\ldots + w_n^2 e_n^2}{(w_1 + w_2 + \ldots\ldots + w_n)^2}$$

$$= e^2 \,.\, \frac{w_1^2 + w_2^2 + \ldots\ldots + w_n^2}{(w_1 + w_2 + \ldots\ldots + w_n)^2} \text{ since the } e\text{'s are equal.}$$

If we make this a minimum with regard to w_1 we have

$$2w_1 (w_1 + w_2 + \ldots + w_n)^2 - 2 (w_1 + w_2 + \ldots + w_n) (w_1^2 + w_2^2 + \ldots + w_n^2) = 0$$

so that $\dfrac{w_1}{w_1^2 + w_2^2 + \ldots + w_n^2} = \dfrac{1}{w_1 + w_2 + \ldots + w_n}$.

Making the same expression a minimum with regard to w_2 we get

$$\frac{w_2}{w_1^2 + w_2^2 + \ldots + w_n^2} = \frac{1}{w_1 + w_2 + \ldots + w_n}$$

&c. &c.

Therefore $w_1 = w_2 = etc. = w_n$, and the *p.e.* of the result $= \sqrt{\dfrac{e^2}{n}} = \dfrac{e}{\sqrt{n}}$.

The actual error of the result is $\dfrac{E_1 + E_2 \ldots + E_n}{n}$ so that the arithmetical mean of the observations is the best value that can be adopted.

X.—*Combination of measures not all equally good.*

As before

$$(p.e. \text{ of result})^2 = \frac{w_1^2 e_1^2 + w_2^2 e_2^2 + \ldots + w_n^2 e_n^2}{(w_1 + w_2 + \ldots + w_n)^2}.$$

If we make this a minimum with regard to w_1 we get

$$2w_1 e_1^2 - \frac{2 \cdot (w_1^2 e_1^2 + w_2^2 e_2^2 + \ldots + w_n^2 e_n^2)}{w_1 + w_2 + \ldots + w_n} = 0$$

that is

$$\frac{w_1 e_1^2}{w_1^2 e_1^2 + w_2^2 e_2^2 + \ldots + w_n^2 e_n^2} = \frac{1}{w_1 + w_2 + \ldots + w_n}.$$

Similarly making the expression a minimum with regard to w_2 we get

$$\frac{w_2 e_2^2}{w_1^2 e_1^2 + w_2^2 e_2^2 + \ldots + w_n^2 e_n^2} = \frac{1}{w_1 + w_2 + \ldots + w_n}$$

&c. &c.

so that $w_1 e_1^2 = w_2 e_2^2 = \text{etc.} = w_n e_n^2 = c$ (say)

or $w_1 = \dfrac{c}{e_1^2},\ w_2 = \dfrac{c}{e_2^2}$, etc. $w_n = \dfrac{c}{e_n^2}$;

therefore $(p.e. \text{ of result})^2 = \dfrac{c\ (w_1 + w_2 + \ldots + w_n)}{(w_1 + w_2 + \ldots + w_n)^2}$

$$= \frac{c}{w_1 + w_2 + \ldots + w_n}$$

or
$$\frac{1}{(p.e.\text{ of result})^2} = \frac{w_1}{c} + \frac{w_2}{c} + \ldots + \frac{w_n}{c}$$
$$= \frac{1}{e_1^2} + \frac{1}{e_2^2} + \ldots + \frac{1}{e_n^2}.$$

Let $\frac{1}{(p.e.)^2}$ be called the *Theoretical Weight.*

Then the theoretical weight of $E_1 = \frac{1}{(p.e.\text{ of } E_1)^2} = \frac{1}{e_1^2} = \frac{w_1}{c}$.

Therefore the "theoretical weight" is proportional to the "combination weight", and the theoretical weight of the final result is equal to the sum of the theoretical weights of the separate measures.

XI.—*Deduction of the value of an angle and of its weight from its several measures.*

Suppose that an angle is observed on a number of zeros and several times on each. The result is subject to errors of "observation" and errors of "graduation".

Let o be the error of mean square of observation for a single observdation

" g " " graduation " "

then (total error of mean square)$^2 = o^2 + g^2$,

and if there were n values at a graduation

(total error of mean square of the mean of this graduation)$^2 = \frac{o^2}{n} + g^2$.

The difficulty is to find o and g. The latter should be found from a great number of observations with the same instrument and this result always used, but this is often inconvenient. The method adopted in the Survey Department is to take each zero mean out, subtract this from the separate observations of that mean, take the sum of the squares of all such results for all the zeros to give the *e.m.s.* of observation. Then take the mean of the zero means and subtract this from each zero mean and take the sum of the squares of the results to give the *e.m.s.* of graduation.

The last equation will give the *e.m.s.* of the mean of each zero. From this we get the weight of each zero, and the sum of the zero weights, which will be the weight of the final mean result.

Now suppose that there are m zeros, and that the values at each zero are

$$a_1\ a_2\ a_3\ \ldots\ldots\ldots\ a_{n'}$$
$$b_1\ b_2\ b_3\ \ldots\ldots\ldots\ b_{n''}$$
&c. &c.

and let the total number be N.

Then we take the mean $A = \dfrac{a_1 + a_2 + a_3 + \ldots + a_{n'}}{n'}$

" " " $B = \dfrac{b_1 + b_2 + b_3 + \ldots + b_{n''}}{n''}$

&c. &c.

and let the mean of A, B, etc. be P.

Subtract A from each of the a's, getting $a_1 - A,\ a_2 - A \ldots a_{n'} - A$

" B " " b's " $b_1 - B,\ b_2 - B \ldots b_{n''} - B$

&c. &c.

then (error of mean square of observation for a single observation)^2

$$= \frac{(a_1 - A)^2 + (a_2 - A)^2 \ldots + (b_1 - B)^2 + (b_2 - B)^2 + \ldots + \&c.}{N - 1} = L^2$$

and (error of mean square of graduation for a single measure)^2

$$= \frac{(A - P)^2 + (B - P)^2 + \ldots + \&c.}{m - 1} \quad \ldots\ldots\ldots\ldots = Q^2$$

Therefore (total error of mean square of a single zero of n observations)^2

$$= \frac{L^2}{n} + Q^2$$

The reciprocal of this is the weight of each zero, and the sum of the weights is the weight of the final result as is shown in Section X.

Now if w_A is weight of A, w_B of B, etc. etc., then the proper value to take for the final result is

$$\frac{w_A\,A + w_B\,B + w_C\,C + \ldots\ldots}{w_A + w_B + w_C + \ldots\ldots}.$$

But we have taken P, therefore the correction to P is

$$\frac{w_A\,A + w_B\,B + w_C\,C + \ldots}{w_A + w_B + w_C + \ldots} - P.$$

In practice for simplicity of work this is transformed as follows:—

Let w_K be the least of the w's,

$$\text{then correction} = \frac{w_A\,(A - P) + w_B\,(B - P) + \ldots + w_K\,(K - P) + \ldots}{w_A + w_B + \ldots + w_K + \ldots}$$

$$= \frac{(w_A - w_K)(A-P) + (w_B - w_K)(B-P) + \ldots + (w_K - w_K)(K-P) + \ldots}{w_A + w_B + \ldots + w_K + \ldots}$$

$$+ \frac{w_K(A-P) + w_K(B-P) + \ldots}{w_A + w_B + \ldots + w_K + \ldots}$$

$$= \frac{(w_A - w_K)(A-P) + (w_B - w_K)(B-P) + \ldots + (w_K - w_K)(K-P) + \ldots}{w_A + w_B + \ldots + w_K + \ldots}$$

XII.—*Method of Least Squares.*

If we have a series of errors x, y, z, etc. arising from different causes, and we want to find their most probable values we proceed as follows :—

We know that the probability of x occurring is $Ke^{-\frac{x^2}{c_1^2}}\delta x$,

" " " y " $K'e^{-\frac{y^2}{c_2^2}}\delta y$,

" " " z " $K''e^{-\frac{z^2}{c_3^2}}\delta z$.

&c. &c.

Therefore the probability of these occurring together is

$$= K\,K'\,K'' \ldots e^{-\frac{x^2}{c_1^2} - \frac{y^2}{c_2^2} - \frac{z^2}{c_3^2} \ldots}\delta x\,\delta y\,\delta z \ldots$$

We get the most probable values of x, y, z, etc. when this is a maximum or when

$$\frac{x^2}{c_1^2} + \frac{y^2}{c_2^2} + \frac{z^2}{c_3^2} + \text{etc. is a minimum.}$$

$$\text{But } c_1^2 = (\text{probable error of } x)^2 \times (2{\cdot}096665)^2,$$

or

$$c_1^2 = \frac{1}{w_1} \times \text{some constant } F.$$

Similarly

$$c_2^2 = \frac{1}{w_2} \times F.$$

Therefore the most probable values of $x, y, z,$ etc. are got by making

$$w_1 x^2 + w_2 y^2 + \text{etc.} \quad \text{a minimum.}$$

This is called the *Method of LEAST SQUARES.*

XIII.—*Method of Least Squares applied to the solution of equations.*

Suppose we have a series of equations:—

$$\left.\begin{array}{l} a_1 x_1 + a_2 x_2 + \dots + a_t x_t \quad = e_a \\ b_1 x_1 + b_2 x_2 + \dots + b_t x_t \quad = e_b \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ n_1 x_1 + n_2 x_2 + \dots + n_t x_t \quad = e_n \end{array}\right\} \text{where } t \text{ is greater than } n,$$

so that we have more unknowns than equations and we want to find the most probable values of $x_1, x_2, \dots x_t$. This is done as shown above by making

$$U = \frac{x_1^2}{u_1} + \frac{x_2^2}{u_2} + \dots \frac{x_t^2}{u_t} \text{ a minimum,}$$

where u_1, u_2, etc. are the reciprocals of the weights of x_1, x_2, etc.

Differentiate all these equations, and we get:—

$$a_1\, dx_1 + a_2\, dx_2 + \dots + a_t\, dx_t = 0$$

$$b_1\, dx_1 + b_2\, dx_2 + \dots + b_t\, dx_t = 0$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$n_1\, dx_1 + n_2\, dx_2 + \dots + n_t\, dx_t = 0$$

$$\frac{x_1\, dx_1}{u_1} + \frac{x_2\, dx_2}{u_2} + \dots + \frac{x_t\, dx_t}{u_t} = 0$$

Multiply the first of these equations by λ_a, the second by λ_b, and so on, and the last by -1 and add them all together.

Then

$$\left\{\lambda_a a_1 + \lambda_b b_1 + \dots \lambda_n n_1 - \frac{x_1}{u_1}\right\} dx_1 + \left\{\lambda_a a_2 + \lambda_b b_2 + \dots + \lambda_n n_2 - \frac{x_2}{u_2}\right\} dx_2 +$$

$$\dots + \left\{\lambda_a a_t + \lambda_b b_t + \dots + \lambda_n n_t - \frac{x_t}{u_t}\right\} dx_t \quad = 0$$

Now as dx_1, dx_2, etc. are all independent, the coefficient of each of them must be equal to zero.

$$\text{Therefore}\quad \left.\begin{aligned} x_1 &= u_1 \{ \lambda_a a_1 + \lambda_b b_1 + \ldots\ldots + \lambda_n n_1 \} \\ x_2 &= u_2 \{ \lambda_a a_2 + \lambda_b b_2 + \ldots\ldots + \lambda_n n_2 \} \\ &\ldots \quad \ldots \quad \ldots \\ x_t &= u_t \{ \lambda_a a_t + \lambda_b b_t + \ldots\ldots + \lambda_n n_t \} \end{aligned}\right\} \quad \ldots \quad (1).$$

Substitute these values in the original n equations and we get:—

$$a_1 u_1 \{ \lambda_a a_1 + \lambda_b b_1 + \ldots + \lambda_n n_1 \} + a_2 u_2 \{ \lambda_a a_2 + \lambda_b b_2 + \ldots + \lambda_n n_2 \}$$

$$+ \ldots\ldots + a_t u_t \{ \lambda_a a_t + \lambda_b b_t + \ldots + \lambda_n n_t \} \quad = e_a$$

&c. &c.

$$\text{Therefore}\quad \lambda_a \{ a_1 a_1 u_1 + a_2 a_2 u_2 + \ldots + a_t a_t u_t \}$$

$$+ \lambda_b \{ a_1 b_1 u_1 + a_2 b_2 u_2 + \ldots + a_t b_t u_t \} + \ldots$$

$$\ldots + \lambda_n \{ a_1 n_1 u_1 + a_2 n_2 u_2 + \ldots + a_t n_t u_t \} \quad = e_a$$

&c. &c.

That is,

$$\left.\begin{aligned} [a\,a\,u]\,\lambda_a + [a\,b\,u]\,\lambda_b + \ldots + [a\,n\,u]\,\lambda_n &= e_a \\ [a\,b\,u]\,\lambda_a + [b\,b\,u]\,\lambda_b + \ldots + [b\,n\,u]\,\lambda_n &= e_b \\ \ldots \qquad \ldots \qquad \ldots & \\ [a\,n\,u]\,\lambda_a + [b\,n\,u]\,\lambda_b + \ldots + [n\,n\,u]\,\lambda_n &= e_n \end{aligned}\right\}$$

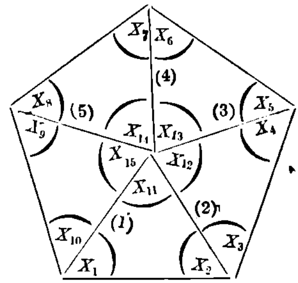
where [ ] represents the summations shown above.

It will be seen from these equations that coefficients in the first horizontal line are the same as those in the first vertical line; the coefficients in the second horizontal line the same as those in the second vertical line and so on, so that in writing down the equations it is usual to omit the quantities below the diagonal.

We have now n equations to solve for the n unknowns λ_a, λ_b . . λ_n. Having got the λ's, then x_1, x_2, &c. x_t are at once obtained from equations (1).

XIV.—*Corrections to the observed angles of a polygon.*

Suppose for example that all the angles in a pentagon are observed and that the errors are $x_1, x_2 \ldots\ldots x_{15}$, so that the true values of the angles are $X_1 - x_1$, $X_2 - x_2 \ldots X_{15} - x_{15}$.



The equations of condition are:—

1. The sum of the observed angles of each triangle must be equal to two right angles, after being corrected for spherical excess, but as this never occurs there is an error, *e*.

Now

$$X_1 - x_1 + X_2 - x_2 + X_{11} - x_{11}$$

$$= 180^\circ + \text{spherical excess}$$

therefore

$$x_1 + x_2 + x_{11} = \text{sum of observed angles} - (180^\circ + \text{spherical excess}) = e_1$$

so that

$$x_1 + x_2 + x_{11} = e_1$$
$$x_3 + x_4 + x_{12} = e_2$$
$$\text{etc.} \qquad \text{etc.}$$

These are the *TRIANGULAR EQUATIONS.*

2. The sum of the observed angles at the central point must be 360°, and as there is also an error here, we have

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = e_6$$

This is the *CENTRAL EQUATION.*

3. If we start at any side and work round through the triangles till we reach that side again we ought to get the value we started with, but as this never occurs there is an error.

This is the *SIDE EQUATION.*

To form it we have as just stated

$$\frac{\text{Sin}\,(X_1 - x_1)}{\text{Sin}\,(X_2 - x_2)}\,\frac{\sin\,(X_3 - x_3)}{\sin\,(X_4 - x_4)}\,\cdots\cdots\,\frac{\sin\,(X_9 - x_9)}{\sin\,(X_{10} - x_{10})}$$

$$= \frac{\sin\,(2)}{\sin\,(1)}\,\frac{\sin\,(3)}{\sin\,(2)}\,\cdots\cdots\cdots\,\frac{\sin\,(1)}{\sin\,(5)} = 1$$

or taking logs.

$$\log_{10} \sin\,(X_1 - x_1) - \log_{10} \sin\,(X_2 - x_2) + \text{etc.} = 0.$$

But $\sin(X_1 - x_1) = \sin X_1 \cos x_1 - \cos X_1 \sin x_1$

$$= \sin X_1 (\cos x_1 - \cot X_1 \sin x_1)$$

$$= \sin X_1 (1 - x_1 \cot X_1 \sin 1'') \text{ since } x_1 \text{ is small}$$

that is, $\sin(X_1 - x_1) = \sin X_1 (1 - a_1 x_1 \sin 1'')$ where $a_1 = \cot X_1$.

But $\log_e (1 - x) = -x - \frac{x^2}{2} - \text{etc.}$

$$= -x \qquad \text{when } x \text{ is small}$$

and $\log_{10}(1 - x) = \log_e (1 - x) \log_{10} e$

$$= -Mx$$

where M is the modulus of the common system of logs. *viz.* ·43429448.

Therefore $\log_{10} \sin(X_1 - x_1) = \log_{10} \sin X_1 + \log_{10}(1 - a_1 x_1 \sin 1'')$

$$= \log_{10} \sin X_1 - Ma_1 x_1 \sin 1''.$$

Similarly for the others.

Therefore $\log_{10} \sin X_1 - Ma_1 x_1 \sin 1'' - \log_{10} \sin X_2 + Ma_2 x_2 \sin 1''$
$+ \log_{10} \sin X_3 - Ma_3 x_3 \sin 1'' - \text{etc.} \qquad = 0.$

Therefore $M \{a_1 x_1 - a_2 x_2 + a_3 x_3 - \text{etc.}\} \sin 1''$

$$= \log_{10} \sin X_1 - \log_{10} \sin X_2 + \log_{10} \sin X_3 - \text{etc.}$$

$$= \log_{10} \left(\frac{\sin X_1 \sin X_3 \dots\dots}{\sin X_2 \sin X_4 \dots\dots} \right)$$

or $$a_1 x_1 - a_2 x_2 + a_3 x_3 - \text{etc.} = \left[\log_{10} \frac{\sin X_1 \sin X_3 \text{ etc.}}{\sin X_2 \sin X_4 \text{ etc.}} \right] \frac{\operatorname{cosec} 1''}{M}.$$

The cotangents a_1, a_2 etc. were formerly used, but now another transformation is used as follows :—

$$\log_{10} \sin X_1 = M \log_e \sin X_1$$

therefore $$\frac{d}{dX_1} \log_{10} \sin X_1 = M \frac{d}{dX_1} \log_e \sin X_1$$

$$= M \frac{\cos X_1}{\sin X_1} = M \cot X_1 = Ma_1$$

Now if dX_1 = circular measure of $1'' = \sin 1''$

then $\qquad s_1 = \text{change of } \log_{10} \sin X_1 \text{ for } 1'' = Ma_1 \sin 1''$

so that the equation becomes

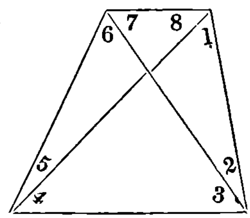
$$s_1 x_1 - s_2 x_2 + s_3 x_3 - \ldots \text{ etc.} = \log_{10} \frac{\sin X_1 \sin X_3 \text{ \&c.}}{\sin X_2 \sin X_4 \text{ \&c.}}$$

$$= e_7.$$

Thus all the equations between the errors are of the first degree and the method of solution is that already given.

Now let us make up the equations for a quadrilateral.

The triangular equations are



$$x_1 + x_2 + x_3 + x_4 = e_1$$

$$x_3 + x_4 + x_5 + x_6 = e_2$$

$$x_5 + x_6 + x_7 + x_8 = e_3$$

the other equation can be neglected, as it is the sum of the first and third minus the second.

The side equation is

$$- s_1 x_1 + s_{2,\,3} (x_2 + x_3) - s_3 x_3 + s_6 x_6 - s_{6,\,7} (x_6 + x_7) + s_8 x_8 = e_4$$

where $\qquad s_{m,\,n} = \text{change in log sin } (X_m + X_n) \text{ for } 1'',$

which are the equations in the computation form.

We have shown above that we get n equations to find n unknowns $\lambda_a, \lambda_b \ldots \lambda_n$. The solution can of course be effected in many ways, but it is better always to follow one method and the one first used by Gauss and now adopted in the Survey of India is as follows:—Multiply the first equation by the coefficient of λ_b and divide by the coefficient of λ_a and record the result with the sign changed. Multiply the first equation by the coefficient of λ_c, divide by the coefficient of λ_a and record the result with the sign changed and so on. Then add second line of the first set of equations to the first line of this set, the third of the first set to the second of the new set and so on. The result will be $n-1$ equations containing $\lambda_b, \lambda_c \ldots \lambda_n$. Proceed in a similar way with this set and we get $n-2$ equations containing $\lambda_c \ldots \lambda_n$ and so on. Finally we get a single equation in λ_n and working backwards we get $\lambda_{n-1} \ldots \lambda_c, \lambda_b, \lambda_a$.

XV.—*Least number of equations in a figure.*

The manner of forming the equations is as follows. Choosing any side of the figure, it does not matter which, as a base, a skeleton figure is drawn consisting of sufficient sides *only* to fix all the stations; every triangle thus formed, of which the three angles have been observed, furnishes a *triangular* equation; the remaining sides are then introduced in succession, and, as each is drawn, the corresponding equations are formed; every side so added introduces one or more observed angles that were not employed before, and each angle furnishes a new equation of condition, usually either a *triangular* or a *side* or a *central* equation; but it may happen that the new side completes, not a triangle, but a four or more sided figure, in which case the triangular equation is merely replaced by one of another kind, usually the geometrical equation between the interior angles of the figure so completed. If, as each angle is introduced, an equation is written down which includes it, no redundant equations will occur.

Let N be the number of observed angles,

„ S „ stations of observation,

then $S - 2$ triangles are required to fix the relative positions of S stations; if of these only P triangles have the three angles observed, there will be $2\ (S-2) + P$ angles giving P triangular equations; every new angle which does not fix a new station, now introduced, gives an additional equation; consequently the number of *new* equations is $N - \{2\ (S - 2) + P\}$, and the total number of equations $= P + N - \{2\ (S-2) + P\} = N - 2\ S + 4$.

Besides the triangular, central and side equations already spoken of, there may be another class of equations called toto-partial equations. These occur when a whole angle has been observed as well as its separate parts. The first thing is to find all the possible triangular equations, central equations and toto-partial equations, and the number of side equations will be the difference between these and the total number above given.

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DEHRA DÚN:
10th February 1903.

Survey of India.

NOTES

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THEORY OF ERRORS OF OBSERVATION.